B.Sc Part – I

Riemannian Christoffel's Tensor

Let A_i be a covariant vector

write

$$A_{i,j} = A_{ij} , A_{ij,k} = A_{ijk} , \cancel{A_{ij,i}}$$

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - A_a \Gamma^a_{ij} = A_{ij}$$

$$A_{ij,k} = \frac{\partial A_{ij}}{\partial x^k} - A_{aj} \Gamma^a_{ik} - A_{ia} \Gamma^a_{jk}$$

$$= \frac{\partial}{\partial x^k}\left(\frac{\partial A_i}{\partial x^j} - A_a \Gamma^a_{ij}\right) - \Gamma^a_{ik}\left(\frac{\partial A_a}{\partial x^j} - A_b \Gamma^b_{aj}\right) -$$

$$\Gamma^a_{jk}\left[\frac{\partial A_i}{\partial x^a} - A_b \Gamma^b_{ia}\right]$$

$$= \frac{\partial^2 A_i}{\partial x^j \partial x^k} - \Gamma^a_{ij}\frac{\partial A_a}{\partial x^k} - A_a \frac{\partial \Gamma^a_{ij}}{\partial x^k} - \Gamma^a_{ik}\frac{\partial A_a}{\partial x^j}$$

$$+ A_b \Gamma^b_{aj} \Gamma^a_{ik} - \Gamma^a_{jk}\frac{\partial A_i}{\partial x^a} + A_b \Gamma^b_{ia} \Gamma^a_{jk}$$

Re-arranging the Terms

$$A_{ij,k} = \left(\frac{\partial^2 A_i}{\partial x^j \partial x^k} - \Gamma^a_{jk}\frac{\partial A_i}{\partial x^a} + A_b \Gamma^a_{ia} \Gamma^a_{jk}\right)$$

$$- \Gamma^a_{ij}\frac{\partial A_a}{\partial x^k} - \Gamma^a_{ik}\frac{\partial A_a}{\partial x^j}$$

$$- A_a \frac{\partial \Gamma^a_{ij}}{\partial x^k} + A_b \Gamma^a_{ik} \Gamma^b_{aj}$$

(1)

Interchanging J and K in this equation

$$A_{i;K,J} = \left[\frac{\partial^2 A_i}{\partial x^K \partial x^J} - \Gamma^a_{KJ}\frac{\partial A_i}{\partial x^a} + A_b \Gamma^b_{ia}\Gamma^a_{KJ}\right.$$

$$- \Gamma^a_{iK}\frac{\partial A_a}{\partial x^J} - \Gamma^a_{iJ}\frac{\partial A_a}{\partial x^K} - A_a \frac{\partial \Gamma^a_{iK}}{\partial x^J} +$$

$$A_b \Gamma^a_{iJ}\Gamma^b_{aK} \quad \text{—— (2)}$$

Subtracting (2) from (1)

$$A_{iJ,K} - A_{iK,J} = -A_a \frac{\partial \Gamma^a_{iJ}}{\partial x^K} + A_b \Gamma^a_{iK}\Gamma^b_{aJ}$$

$$+ A_a \frac{\partial \Gamma^a_{iK}}{\partial x^J} - A_b \Gamma^a_{iJ}\Gamma^b_{aK}$$

$$A_{iJ,K} - A_{iK,J} = A_a\left[-\frac{\partial \Gamma^a_{iJ}}{\partial x^K} + \frac{\partial \Gamma^a_{iK}}{\partial x^J} - \Gamma^b_{iJ}\Gamma^a_{bK}\right.$$

$$\left. + \Gamma^b_{iK}\Gamma^a_{bJ}\right]$$

Taking $R^a_{iJK} = -\frac{\partial \Gamma^a_{iJ}}{\partial x^K} + \frac{\partial \Gamma^a_{iK}}{\partial x^J} - \Gamma^b_{iJ}\Gamma^a_{bK}$

$$+ \Gamma^b_{iK}\Gamma^a_{b}$$

we ~~get~~

$$A_{iJ,K} - A_{iK,J} = A_a R^a_{iJK} \quad \text{—— (3)}$$